bl2086 的問題

About **PDE**:
Walter Strauss, Partial Differential Equations, An Introduction, 2nd Edition, P.47
5. Prove properties (a) to (e) of the diffusion equation (1)($u_t = ku_{xx}$).
(a) The translate $u(x-y,\ t)$ of any solution $u(x,\ t)$ is another solution, for any fixed y.
SOLUTION:
Let $v(x,\ t) = u(x-y_0,\ t)$, For $u_t = ku_{xx}$, $v_t = u_t(x-y_0,\ t) = ku_{xx}(x-y_0,\ t)$, $v_x = u_x(x-y_0,\ t)$, $v_{xx} = u_{xx}(x-y_0,\ t)$, Then $v_t = u_t(x-y_0,\ t) = ku_{xx}(x-y_0,\ t) = kv_{xx}$ Q.E.D.

(b) Any derivative (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.
SOLUTION:
$(u_x)_t = u_{tx} = (u_t)_x = (ku_{xx})_x = ku_{xxx} = k(u_x)_{xx}$
$(u_t)_t = (ku_{xx})_t = ku_{txx} = k(u_t)_{xx}$
$(u_{xx})_t = u_{txx} = (u_t)_{xx} = (ku_{xx})_{xx} = ku_{xxxx} = k(u_{xx})_{xx}$ Q.E.D.

(c) A linear combination of solutions of (1) is again a solution of (1)($u_t = ku_{xx}$).(This is just linearity.)
SOLUTION:
For $u_t = ku_{xx}$, Let $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$, then $u_t = c_1(u_1)_t + c_2(u_2)_t + \cdots + c_n(u_n)_t = k(c_1(u_1)_{xx} + c_2(u_2)_{xx} + \cdots + c_n(u_n)_{xx}) = ku_{xx}$ Q.E.D.

(d)An integral of solutions is again a solution. Thus if $S(x,t)$ is a solution of (1)($u_t = ku_{xx}$), then so is $S(x-y,\ t)$ and so is

$$v(x,\ t) = \int_{-\infty}^{\infty} S(x-y,\ t)g(y)dy$$

for any function g(y), as long as this improper integral converges appropri-

ately. (We' ll worry about convergence later.) In fact, (d) is just a limiting form of (c).
SOLUTION:
For $S_t(x-y,\ t) = kS_{xx}(x-y,\ t)$

$$v_t(x,\ t) = \int_{-\infty}^{\infty} S_t(x-y,\ t)g(y)dy = \int_{-\infty}^{\infty} kS_{xx}(x-y,\ t)g(y)dy$$

$$v_x(x,\ t) = \int_{-\infty}^{\infty} S_x(x-y,\ t)g(y)dy,\ v_{xx}(x,\ t) = \int_{-\infty}^{\infty} S_{xx}(x-y,\ t)g(y)dy$$

$$v_t(x,\ t) = \int_{-\infty}^{\infty} S_t(x-y,\ t)g(y)dy = \int_{-\infty}^{\infty} kS_{xx}(x-y,\ t)g(y)dy$$

$$= k\int_{-\infty}^{\infty} S_{xx}(x-y,\ t)g(y)dy = kv_{xx}(x,\ t)$$

Q.E.D.

(e)If $u(x,\ t)$ is a solution of (1)($u_t = ku_{xx}$), so is the dilated function $u(\sqrt{a}x,\ at)$, for any $a > 0$. Prove this by the chain rule:
SOLUTION:
Let $v(x,\ t) = u(\sqrt{a}x,\ at)$, then $v_t = au_t(\sqrt{a}x,\ at) = aku_{xx}(\sqrt{a}x,\ at)$
$v_x = \sqrt{a}u_x(\sqrt{a}x,\ at)$, and $v_{xx} = au_{xx}(\sqrt{a}x,\ at)$, so $v_t = au_t(\sqrt{a}x,\ at) = aku_{xx}(\sqrt{a}x,\ at) = kv_{xx}$ Q.E.D.